- **5285:** *Proposed by D.M. Bătinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzu, Romania*

Let $\{a_n\}_{n\geq 1}$, and $\{b_n\}_n \geq 1$ be positive sequences of real numbers with $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a\in\Re_+$ and $\lim\limits_{n\to\infty}\dfrac{b_{n+1}}{nb_n}=b\in\Re_+$.
For $x\in\Re$, calculate

$$\lim_{n\to\infty}\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x}-\left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right).$$

Solution 1 by Arkady Alt, San Jose, CA

Since the $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a$, then by the Stolz Theorem $\lim\limits_{n\to\infty}\dfrac{a_n}{n}=a$. Also note that

$$\lim_{n\to\infty}\frac{\dfrac{b_{n+1}}{(n+1)!}}{\dfrac{b_n}{n!}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)\,b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)\,b_n}\cdot\frac{n+1}{n}=\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b.$$

By the Multiplicative Stolz Theorem $\lim\limits_{n\to\infty}\dfrac{\dfrac{b_n}{n!}}{\dfrac{b_{n+1}}{(n+1)!}}=b$ yields $\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{b_n}{n!}}=b$.

$$\text{Let } c_n=\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}=\frac{\sqrt[n+1]{\dfrac{b_{n+1}}{(n+1)!}}}{\sqrt[n]{\dfrac{b_n}{n!}}}\cdot\frac{\dfrac{\sqrt[n+1]{(n+1)!}}{n+1}}{\dfrac{\sqrt[n]{n!}}{n}}\cdot\frac{n+1}{n}.$$

Since $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}=\dfrac{1}{e}$, $\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{b_n}{n!}}=b$, $\lim\limits_{n\to\infty}\dfrac{n+1}{n}=1$ then $\lim\limits_{n\to\infty}c_n=1$, and, therefore, $\lim\limits_{n\to\infty}\dfrac{c_n^{\cos^2 x}-1}{\ln\left(c_n^{\cos^2 x}\right)}=1$.

Hence, $\lim\limits_{n\to\infty}n\left(c_n^{\cos^2 x}-1\right)=\lim\limits_{n\to\infty}\left(n\ln\left(c_n^{\cos^2 x}\right)\cdot\dfrac{c_n^{\cos^2 x}-1}{\ln\left(c_n^{\cos^2 x}\right)}\right)=\lim\limits_{n\to\infty}n\ln\left(c_n^{\cos^2 x}\right)=$

$\cos^2 x\lim\limits_{n\to\infty}n\ln c_n=\cos^2 x\ln\left(\lim\limits_{n\to\infty}c_n^n\right)=\cos^2 x\ln\left(\lim\limits_{n\to\infty}\dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n}\right)$.

Since $\dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n}=\dfrac{b_{n+1}}{nb_n}\cdot\dfrac{1}{\sqrt[n+1]{\dfrac{b_{n+1}}{(n+1)!}}}\cdot\dfrac{n}{\sqrt[n+1]{(n+1)}}$, then $\lim\limits_{n\to\infty}\dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n}=b\cdot\dfrac{1}{b}\cdot e=e$

and, therefore, $\lim\limits_{n\to\infty}n\left(c_n^{\cos^2 x}-1\right)=\cos^2 x$.

And since $a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x}-\left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)=$
$\left(\frac{a_n}{n}\right)^{\sin^2 x}\cdot\left(\sqrt[n]{\frac{b_n}{n!}}\right)^{\cos^2 x}\cdot\left(\frac{\sqrt[n]{n!}}{n}\right)^{\cos^2 x}\cdot n\left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x}-1\right)$ then

$\lim_{n\to\infty}\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x}-\left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right)=$
$\mathrm{a}^{\sin^2 x}b^{\cos^2 x}e^{-\cos^2 x}\lim_{n\to\infty}n\left(c_n^{\cos^2 x}-1\right)=a^{\sin^2 x}b^{\cos^2 x}e^{-\cos^2 x}\cos^2 x.$